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Pbar Note 575
Moment Method Formulation for Beam Excitation of Waveguide Slots

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Introduction

This paper describes a moment method formulation for calculating the pickup and kicker impedances of a stochastic cooling waveguide structure. A schematic of the waveguide pickup is shown in Figure 1.

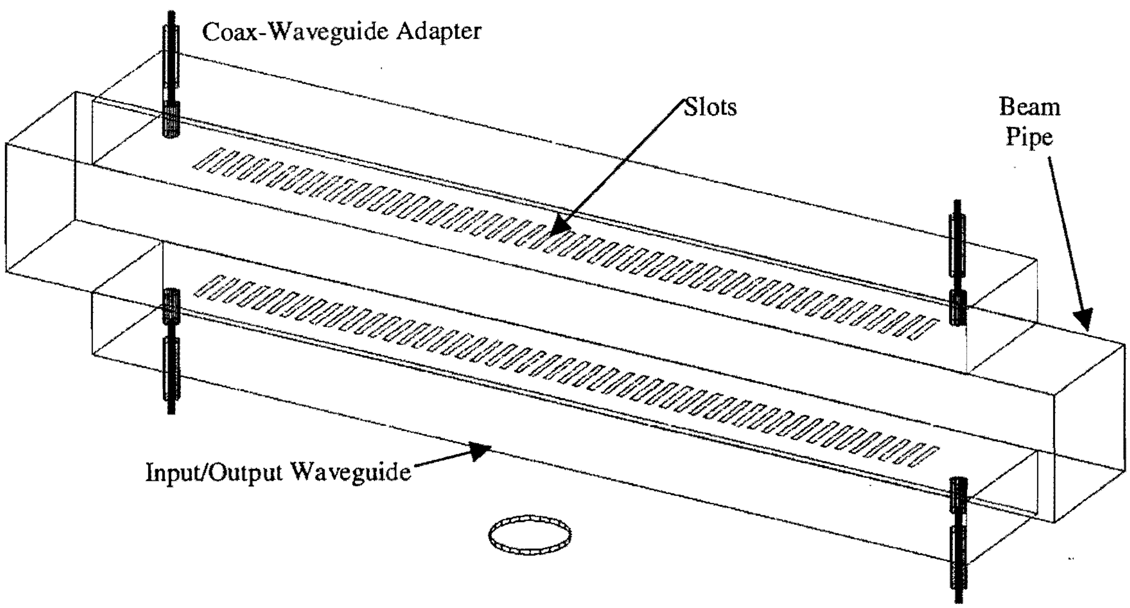

Figure 1. Schematic of a stochastic cooling waveguide pickup/kicker.

Slots carved in a waveguide wall will slow down the phase velocity of a wave in the waveguide. The reduction in phase velocity is a function of the slot length, width, and the spacing between slots. The coupling of the slots to the beam is proportional to the slot length. When the reduced phase velocity of the waveguide matches the beam velocity, the coupling of the slots will add constructively. In this slow-wave mode, the gain of the array is proportional to the number of slots and the bandwidth of the array is inversely proportional to the number of slots.

Finite element methods for are poorly suited for solving electromagnetic problems with thin wall apertures. The thin wall causes the electromagnetic field pattern to vary rapidly in the vicinity of the aperture. For finite elements, this would require a fine mesh around the apertures resulting in very large matrices to invert. Also, finite elements yield the solution for the electromagnetic field everywhere in the problem. To calculate pickup and kicker impedances, the electromagnetic field has to be known only at the slots or along the beam path.

For these reasons, a moment method approach will be used for calculating the pickup and kicker impedances.

Geometry of the Problem

The geometry of the problem is shown in Figures 2 and 3. Two regions (Regions I & II) are separated by a conducting screens in the X-Z plane. These regions may have different dielectric constants or backing plate configurations. The beam travels in the z direction somewhere in Region I. The two regions are connected by a hole or aperture in the screen. The purpose of the moment method program is to find the tangential electric and magnetic fields in this aperture.

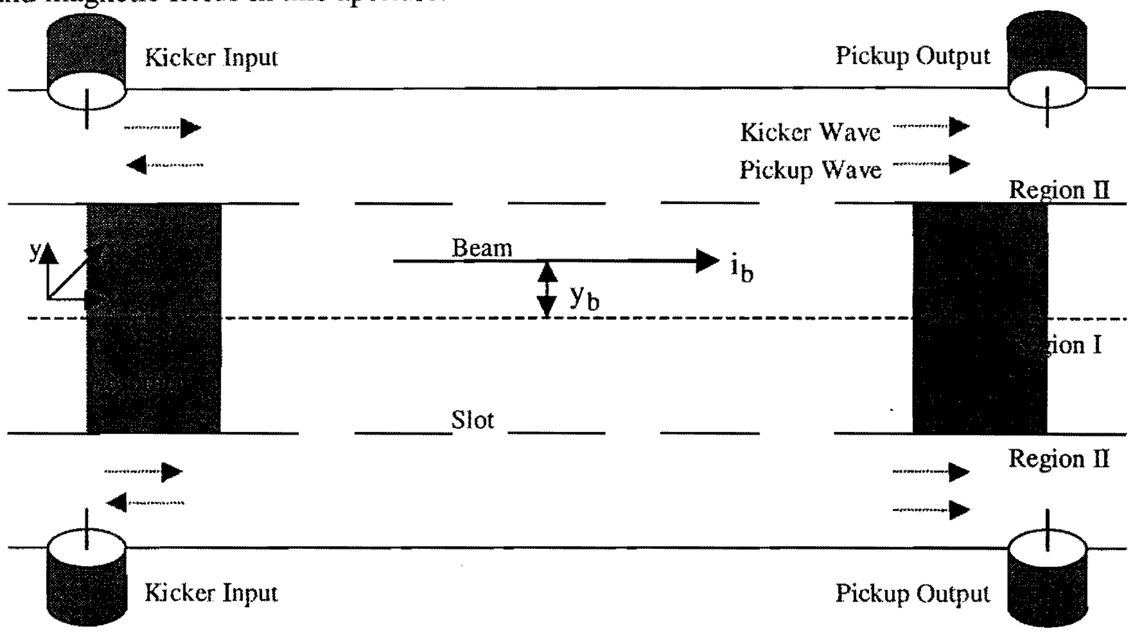

Figure 2. Side view of a stochastic cooling waveguide pickup/kicker

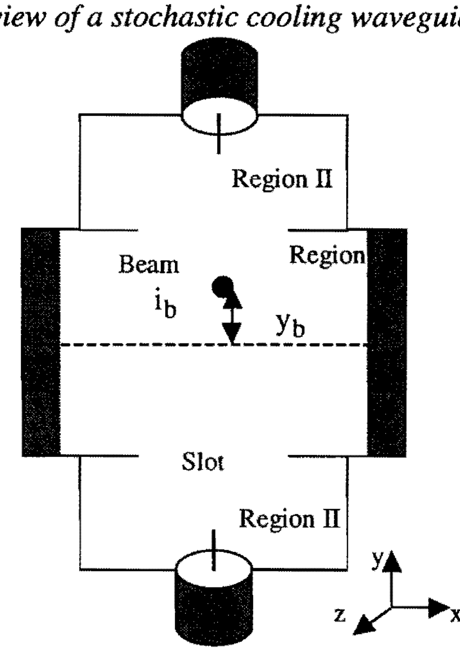

Figure 3. Head-on view of stochastic cooling waveguide pickup/kicker

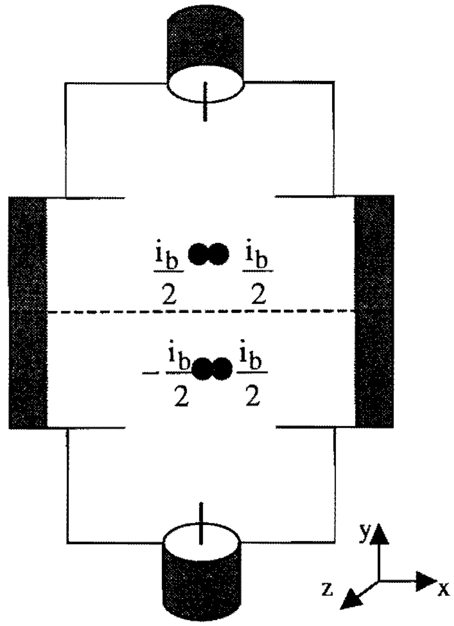

Figure 4. Decomposition of an off-center beam into the sum and difference modes.

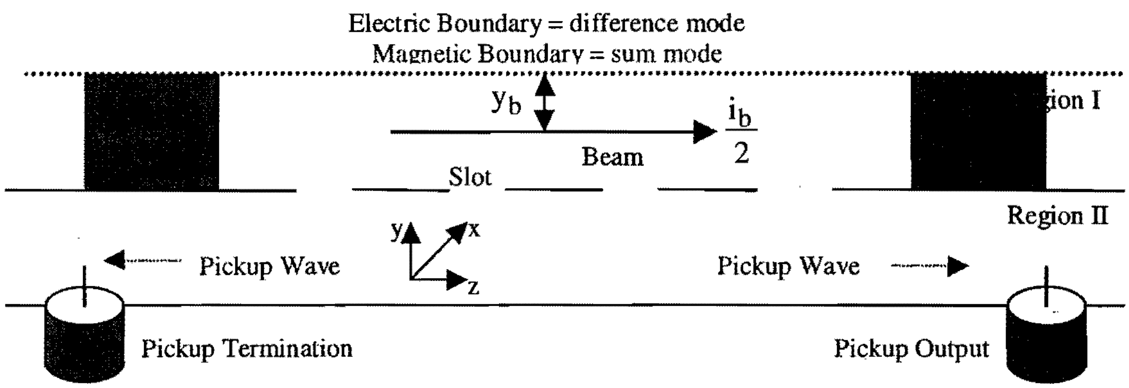

Figure 5. Side-view of ½ of the pickup after symmetry decomposition

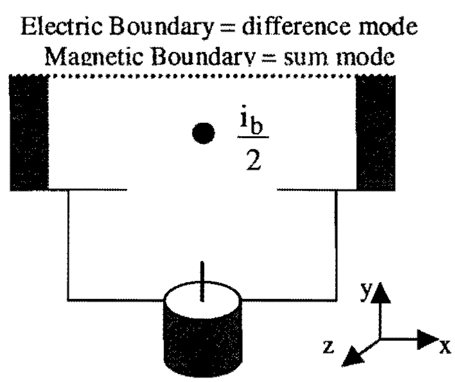

Figure 6. Head-on view of ½ of the pickup after symmetry decomposition.

The problem can be divided into sum and difference modes as shown in Figure 4. The sum and difference modes will be used for momentum and transverse cooling structures, respectively. For the sum mode, the magnetic field component parallel to the x-z plane at the center of the structure is zero. The x-z plane at the center can then be replaced with a magnetic conductor as shown in Figures 5 and 6. For the difference mode, the electric field component parallel to the x-z plane at the center of the structure is zero. The x-z plane at the center can then be replaced with an electric conductor.

Magnetic Current Sources

The moment method approach for this problem will be to solve for the tangential magnetic field in the aperture. Because the tangential electric field is zero on the conducting screen that separates the two regions, this approach is best formulated using magnetic current sources instead of electric current sources. Since magnetic current sources are an unfamiliar topic with most people, this section will describe the properties of magnetic current sources.

Since magnetic charge has not been found to exist, a magnetic current is defined by the Equivalence Principle. As shown in Fig 7a, a set of sources produces a field, E and H. A imaginary boundary is now drawn around the sources. The Equivalence Principle states that the same field, E and H, will exist outside the boundary if there is zero field and no sources inside the boundary but the boundary is coated with equivalent surface currents as shown in Figure 7b. These sources are:

$$\vec{J}_S = \hat{n} \times \vec{H}$$
$$\vec{M}_S = \vec{E} \times \hat{n} \tag{1}$$

where:

$$\vec{\nabla} \times \vec{H} = j\omega\varepsilon\vec{E} + \vec{J}_S\delta(\vec{r} - \vec{r}_S) \tag{2}$$

and:

$$-\vec{\nabla} \times \vec{E} = j\omega\mu\vec{H} + \vec{M}_S\delta(\vec{r} - \vec{r}_S) \tag{3}$$

The time dependence assumed is $e^{j\omega t}$.

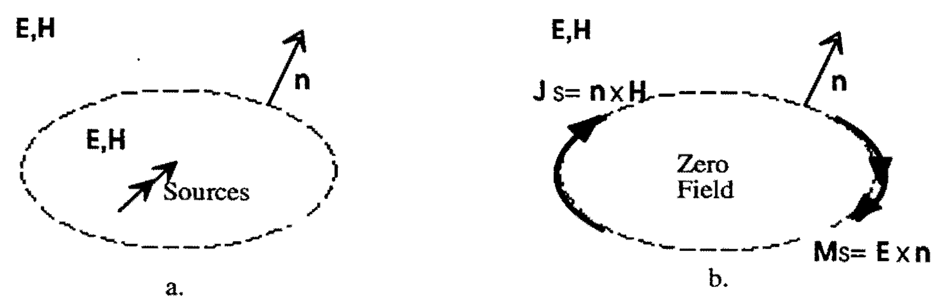

Figure 7.

Since the fields are zero just inside the boundary we can place any material we want inside the boundary and not effect the fields outside of the boundary. In Figure 8a, the boundary is replaced by an electric conductor ($E_{tan} = 0$) which shorts out the electric current J_S. Likewise, in Figure 8b, the boundary is replaced by a magnetic conductor ($H_{tan} = 0$) which shorts out the magnetic current M_S.

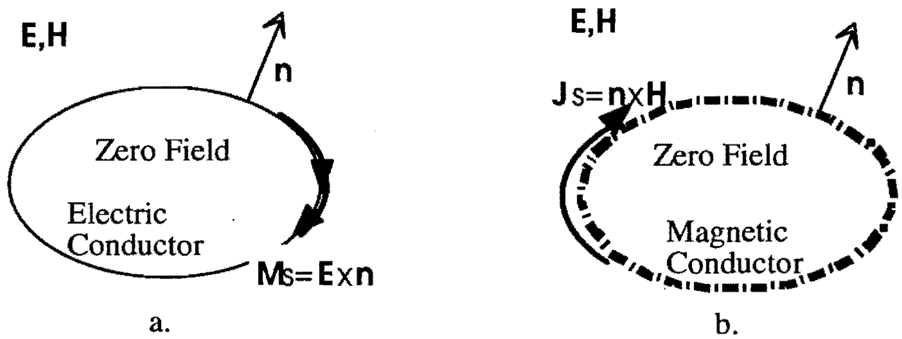

Figure 8.

Consider the case of Figure 3a. Equation 2 reduces to:

$$\vec{\nabla}\times\vec{H} = j\omega\vec{E} \tag{4}$$

From the Continuity equation between electric charge and electric current, if there is no electric current, there is no electric charge. If there is zero electric charge, the divergence of E is zero. When the divergence of E is zero, an electric vector potential can be defined as:

$$\vec{E} = -\frac{1}{\varepsilon}\vec{\nabla}\times\vec{F} \tag{5}$$

With the appropriate choice of gauge, Maxwell's equations can be combined into:

$$\begin{aligned}\left(\nabla^2 + \kappa^2\right)\vec{F} &= -\varepsilon\vec{M}\\ \kappa^2 &= \omega^2\mu\varepsilon\end{aligned} \tag{6}$$

The magnetic field is given as:

$$\vec{H} = -\frac{j\omega}{\kappa^2}\left(\kappa^2\vec{F} + \vec{\nabla}\left(\vec{\nabla}\bullet\vec{F}\right)\right) \tag{7}$$

Equation 6 can be solved by Green's function techniques where:

$$\left(\nabla^2 + \kappa^2\right)G\left(\bar{r}\middle|\bar{r}'\right) = -\varepsilon\delta(\bar{r} - \bar{r}') \tag{8}$$

and:

$$\vec{F}(\bar{r}) = \iiint_V \vec{M}(\bar{r}')G\left(\bar{r}\middle|\bar{r}'\right)dv' \tag{9}$$

If the magnetic current is a surface current as given in Equation 3, then the magnetic field is given as:

$$\vec{H}(\vec{r}) = -\frac{j\omega}{\kappa^2}\left(\kappa^2 \iint_S \vec{M}_s(\vec{r}')G(\vec{r}|\vec{r}')ds' + \vec{\nabla}\iint_S \left(\vec{\nabla}' \bullet \vec{M}_s(\vec{r}')\right)G(\vec{r}|\vec{r}')ds'\right) \tag{10}$$

We will define the right hand side of Equation 10 as an operator on M_S that produces H(r). That is:

$$\vec{H}^{(k)}(\vec{r}) \equiv \vec{H}^{(k)}(\vec{M}_s) \tag{11}$$

where k denotes whether the field is for Region I (k=1) or Region II (k=2).

The Theory of Moment Methods

Using the Equivalence Principle, the fields in Regions I and II of Figure 5 will remain unchanged if the aperture is replaced by conducting screen coated with an equivalent magnetic current source as shown in Figure 9.

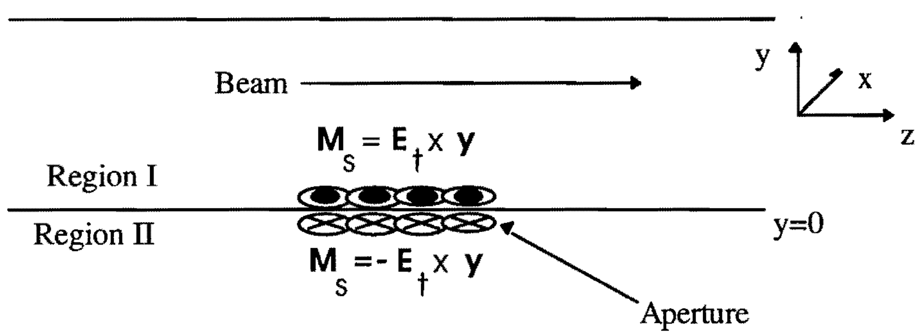

Figure 9.

The magnetic current source in Region I is:

$$\vec{M}_S = \vec{E}_t \times \hat{y} \tag{12}$$

where E_t is the tangential electric field that existed in the aperture before it was replaced with conducting screen. To guarantee continuity of the tangential electric field in the aperture, the equivalent magnetic current source in Region II is:

$$\vec{M}_S = -\vec{E}_t \times \hat{y} \tag{13}$$

The tangential magnetic field in Region I just above the aperture is given as the sum of the incident field due to the beam with the aperture replaced by conductor and the magnetic field due to the equivalent magnetic current source. That is at y=0:

$$\vec{H}_t^{(1)} = \vec{H}_t^{(inc)}{}_1 + \vec{H}_t^{(1)}\left(\vec{E}_t \times \hat{y}\right) \tag{14}$$

It will be assumed that the incident magnetic field can be determined analytically or by other methods. The tangential field in Region II just below the aperture at y=0 is:

$$\vec{H}_t^{(2)} = \vec{H}_t^{(inc)}{}_2 + \vec{\mathcal{H}}_t^{(2)}\left(-\vec{E}_t \times \hat{y}\right) \tag{15}$$

Continuity of the tangential magnetic field through the aperture requires:

$$\vec{H}_t^{(1)} = \vec{H}_t^{(2)} \tag{16}$$

which results in the following equation:

$$\vec{H}_t^{(inc)}{}_2 - \vec{H}_t^{(inc)}{}_1 = \vec{\mathcal{H}}_t^{(1)}\left(\vec{E}_t \times \hat{y}\right) + \vec{\mathcal{H}}_t^{(2)}\left(\vec{E}_t \times \hat{y}\right) \tag{17}$$

This is the key equation of the moment method. Since $H^{(inc)}$ is known, this equation can be inverted to determine E_t in the aperture. Because Equation 17 is an integral equation, it is best solved by numerical methods. Let the tangential electric field in the aperture be given by:

$$\vec{E}_t = \hat{x}\sum_n E_{x_n}\theta_n(x,z) + \hat{z}\sum_n E_{z_n}\psi_n(x,z) \tag{18}$$

where $\theta_n(x,z)$ and $\psi_n(x,z)$ are a set of orthogonal functions. Equation. 17 can be turned into a matrix equation by multiplying it by a set of orthogonal weighting functions $\phi_m(x,z)$ and integrating over the entire x-z plane. The following matrix elements are defined:

$$\left\langle \phi_m \middle| \mathcal{H}_v^{(k)} \middle| \theta_n \right\rangle = \iint_{x,z} \left(\phi_m(x,z)\mathcal{H}_v^{(k)}\left(\hat{z}\theta_n(x,z)\right)\right) dxdz \tag{19}$$

$$\left\langle \phi_m \middle| \mathcal{H}_v^{(k)} \middle| \psi_n \right\rangle = \iint_{x,z} \left(\phi_m(x,z)\mathcal{H}_v^{(k)}\left(\hat{x}\psi_n(x,z)\right)\right) dxdz \tag{20}$$

$$\left\langle \phi_m \middle| H_v^{(inc)}{}_i \right\rangle = \iint_{x,z} \left(\phi_m(x,z) H_v^{(inc)}{}_i(x,z)\right) dxdz \tag{21}$$

Equation 17 becomes:

$$\left\langle \phi_m \middle| H_x^{(inc)}{}_1 \right\rangle - \left\langle \phi_m \middle| H_x^{(inc)}{}_2 \right\rangle = \sum_n \left(\sum_k \left\langle \phi_m \middle| \mathcal{H}_x^{(k)} \middle| \psi_n \right\rangle \right) E_{z_n} - \sum_n \left(\sum_k \left\langle \phi_m \middle| \mathcal{H}_x^{(k)} \middle| \theta_n \right\rangle \right) E_{x_n} \tag{22}$$

$$\left\langle \phi_m \middle| H_z^{(inc)}{}_1 \right\rangle - \left\langle \phi_m \middle| H_z^{(inc)}{}_2 \right\rangle = \sum_n \left(\sum_k \left\langle \phi_m \middle| \mathcal{H}_z^{(k)} \middle| \psi_n \right\rangle \right) E_{z_n} - \sum_n \left(\sum_k \left\langle \phi_m \middle| \mathcal{H}_z^{(k)} \middle| \theta_n \right\rangle \right) E_{x_n} \tag{23}$$

Equations 22 and 23 form a set of linear equations which can be inverted to find the electric field coefficients E_{zn} and E_{xn}. If the electric field expansion functions, ϕ and ψ, are chosen to be as close to the actual solution as possible then only a few terms of the expansion will be needed and the size of the matrix to be inverted will be minimized.

Resistive Terminations

The above derivation ignored resistive terminations in the aperture. In some applications, the signal induced on a slot flows out of the slot to a combiner board by means of microstrip line on the shadow side of the conducting screen as shown in Figure 10. This paper will model the microstrip connection to the slot as a thin film resistor as shown in Figure 11.

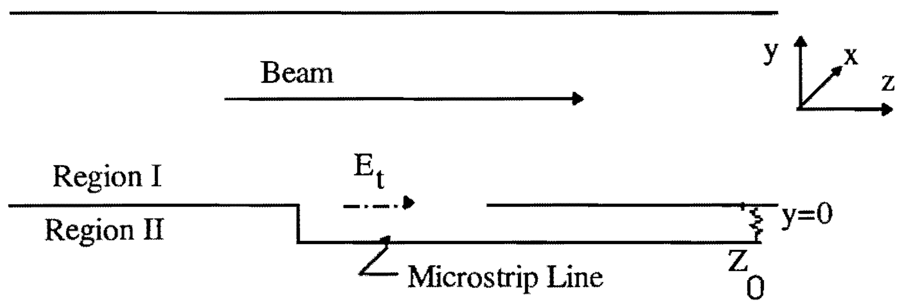

Figure 10.

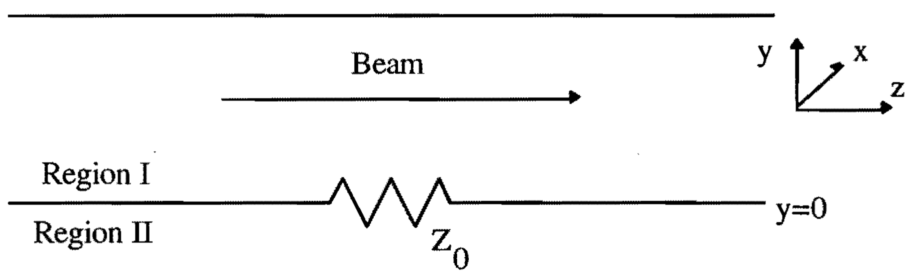

Figure 11. Note that the resistor does not cover the entire aperture in the y direction.

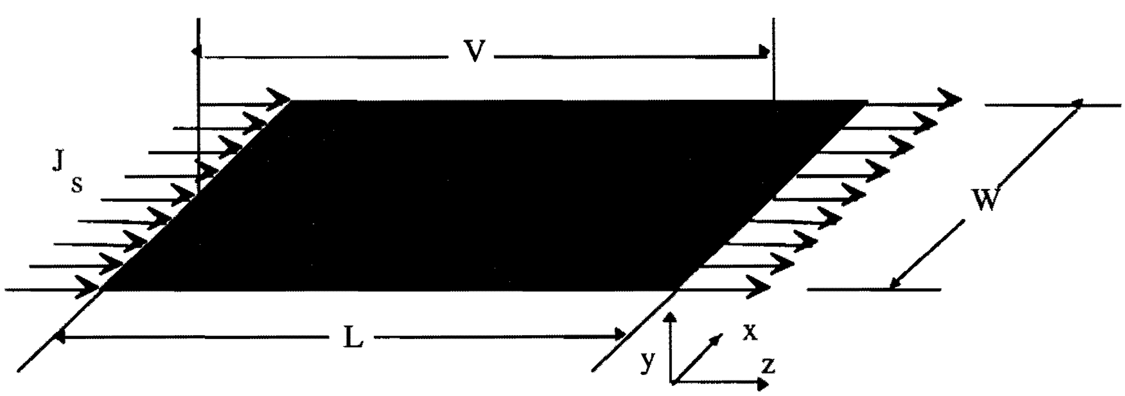

Figure 12.

Consider a thin film resistor shown in Figure 12 with a conductance/square of g Ω^{-1}. The conductance for a uniform sheet of electric current flowing in the z direction is:

$$G = g\frac{W}{L} \tag{24}$$

where W is the width and L is the length of the resistor. Now consider portions of the aperture shown in Figure 5 filled with some of this thin film resistor. Equation 16 becomes:

$$\vec{H}_t^{(1)} - \vec{H}_t^{(2)} = -g(x,z)\cdot\left(\vec{E}_t \times \hat{y}\right) \tag{25}$$

Equation 17 becomes:

$$\vec{H}_t^{(inc)}{}_2 - \vec{H}_t^{(inc)}{}_1 = \vec{\boldsymbol{H}}_t^{(1)}\left(\vec{E}_t \times \hat{y}\right) + \vec{\boldsymbol{H}}_t^{(2)}\left(\vec{E}_t \times \hat{y}\right) + g(x,z)\cdot\left(\vec{E}_t \times \hat{y}\right) \tag{26}$$

Equations 22 and 23 become:

$$\left\langle \phi_m \middle| H_x^{(inc)}{}_1 \right\rangle - \left\langle \phi_m \middle| H_x^{(inc)}{}_2 \right\rangle$$
$$\sum_n \left(\sum_k \langle \phi_m | \boldsymbol{H}_x^{(k)} | \psi_n \rangle + \langle \phi_m | g | \psi_n \rangle \right) E_{z_n} - \sum_n \left(\sum_k \langle \phi_m | \boldsymbol{H}_x^{(k)} | \theta_n \rangle \right) E_{x_n} \tag{27}$$

$$\left\langle \phi_m \middle| H_z^{(inc)}{}_1 \right\rangle - \left\langle \phi_m \middle| H_z^{(inc)}{}_2 \right\rangle =$$
$$\sum_n \left(\sum_k \langle \phi_m | \boldsymbol{H}_z^{(k)} | \psi_n \rangle \right) E_{z_n} - \sum_n \left(\sum_k \langle \phi_m | \boldsymbol{H}_z^{(k)} | \theta_n \rangle + \langle \phi_m | g | \theta_n \rangle \right) E_{x_n} \tag{28}$$

where:

$$\langle \phi_m | g | \psi_n \rangle = \iint_{x,z} \left(\phi_m(x,z)\cdot g(x,z)\cdot \psi_n(x,z) \right) dxdz \tag{29}$$

$$\langle \phi_m | g | \theta_n \rangle = \iint_{x,z} \left(\phi_m(x,z)\cdot g(x,z)\cdot \theta_n(x,z) \right) dxdz \tag{30}$$

Transverse Slot Between Two Waveguides

This section will examine the problem of a transverse coupling slot between two waveguides as shown in Figures 13 and 14. The beam flows in the upper waveguide and the output signal flows out of the lower waveguide. Also the lower waveguide could be the housing for a combiner board network for the slots.

We will confine the slots to lie along the y direction only. Also, the width of the slot (W_i) will be very small compared to the wavelength of excitation. These restrictions will allow us to neglect the y component of electric field in the slot. Also, we will consider the case for an extremely reletavistic beam so that z component of magnetic field in the slots may also be neglected. These assumptions reduce Equations 27 and 28 to:

$$\left\langle \phi_m \middle| H_x^{(inc)}{}_1 \right\rangle - \left\langle \phi_m \middle| H_x^{(inc)}{}_2 \right\rangle = \sum_n \left(\sum_k \langle \phi_m | H_x^{(k)} | \psi_n \rangle + \langle \phi_m | g | \psi_n \rangle \right) E_{z_n} \tag{31}$$

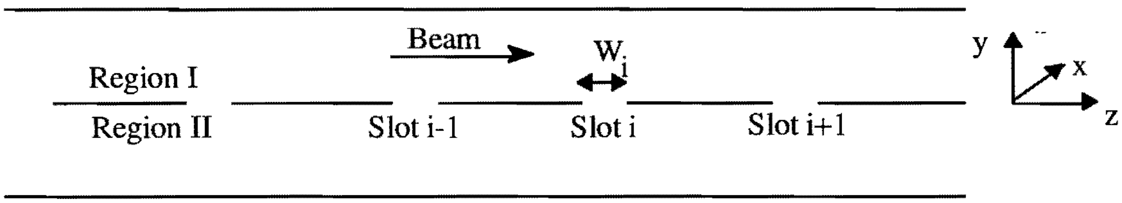

Figure 13. Side long view of coupled waveguide geometry.

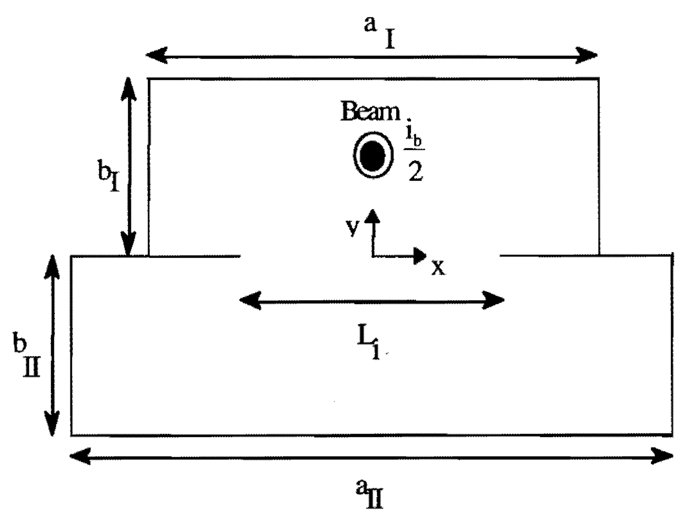

Figure 14. Head on view of coupled waveguide geometry

In the absence of the coupling slots shown in Figure 14, the fields inside the waveguides can be expanded as sum of all the waveguide modes.

$$\vec{E}^+ = \sum_n C_n^+ \left(\hat{e}_{t_n} + \hat{e}_{z_n} \right) e^{-j\beta_n z} \tag{32a}$$

$$\vec{E}^- = \sum_n C_n^- \left(\hat{e}_{t_n} - \hat{e}_{z_n} \right) e^{j\beta_n z} \tag{32b}$$

$$\vec{H}^+ = \sum_n C_n^+ \left(\hat{h}_{t_n} + \hat{h}_{z_n} \right) e^{-j\beta_n z} \tag{32c}$$

$$\vec{H}^- = \sum_n C_n^- \left(-\hat{h}_{t_n} + \hat{h}_{z_n} \right) e^{j\beta_n z} \tag{32d}$$

Consider a source inside a volume v surrounded by a surface S_o in a waveguide as shown in Figure 15.

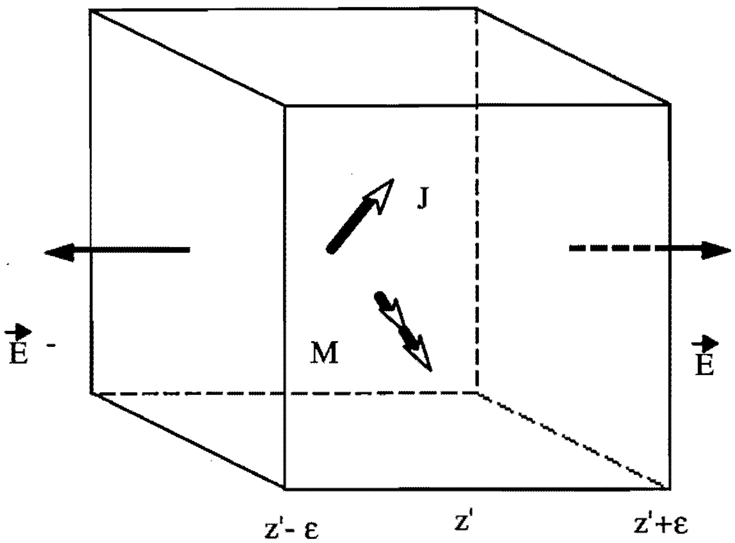

Figure 15. Elemental volume in waveguide containing sources.

For two independent sets of fields and two independent sets of sources, the Lorentz reciprocity theorem states:

$$\oiint_{S_o}\left(\vec{E}^a \times \vec{H}^b - \vec{E}^b \times \vec{H}^a\right)\bullet \hat{n}dS = \iiint_v \left(\vec{E}^a \bullet \vec{J}^b - \vec{H}^a \bullet \vec{M}^b - \vec{E}^b \bullet \vec{J}^a + \vec{H}^b \bullet \vec{M}^a\right)dv \tag{33}$$

Let the **a** field be the field due to the sources and the **b** field be one of the reverse traveling waveguide modes.

$$\vec{E}^a = \vec{E} \qquad \vec{H}^a = \vec{H} \qquad \vec{J}^a = \vec{J} \qquad \vec{M}^a = \vec{M}$$
$$\vec{E}^b = \left(\hat{e}_{t_m} - \hat{e}_{z_m}\right)e^{j\beta_m z} \qquad \vec{H}^b = \left(-\hat{h}_{t_m} + \hat{h}_{z_m}\right)e^{j\beta_m z} \qquad \vec{J}^b = 0 \qquad \vec{M}^b = 0 \tag{34}$$

Substituting Equation 34 into Equation 33 and using the facts that the tangential electric field on the walls of the waveguide are zero and that the waveguide modes are orthogonal, the mode coefficients of Equation 32 for the positive going field are:

$$C_m^+ = \frac{1}{4P_m}\iiint_v \left(\left(\hat{e}_{t_m} - \hat{e}_{z_m}\right)\bullet \vec{J} - \left(-\hat{h}_{t_m} + \hat{h}_{z_m}\right)\bullet \vec{M}\right)e^{j\beta_m z}dv \tag{35}$$

where:

$$P_m = \frac{1}{2}\iint_{S_t}\left(\hat{e}_{t_m} \times \hat{h}_{t_m}\right)\bullet \hat{z}dS \tag{36}$$

If the b field is now set equal to one of the positive going waveguide mode, the mode coeffcients for the negative going field is:

$$C_m^{\pm} = \frac{1}{4P_m} \iiint_V \left(\left(\hat{e}_{t_m} \mp \hat{e}_{z_m} \right) \bullet \bar{J} - \left(\mp \hat{h}_{t_m} + \hat{h}_{z_m} \right) \bullet \vec{M} \right) e^{\pm j\beta_m z} dv \qquad (37)$$

We have already stated that y component of the electric field in the slot will be neglected. Also we will separate the y dependence and the z dependence of the electric field in each slot so that the tangential electric field in all of the slots is written as:

$$\vec{E}_{slots} = \hat{z} \sum_i \lambda_i(z) \left(\sum_l E_{z_{i,l}} \alpha_{i,l}(x) \right) \qquad (38)$$

Where the **I** index indicates the slot number and the **l** index indicates the slot-mode (to be defined later). This is equivalent to specifyfing the expansion function in Equation 18 to be:

$$\psi_n(x,z) = \psi_{i,l}(x,z) = \alpha_{i,l}(x)\lambda_i(z) \qquad (39)$$

For the time being let $\lambda_i(z)$ be equal to the Dirac delta function $\delta(z\text{-}z')$. The magnetic current source due to slot I and slot-mode l in the upper waveguide is:

$$\vec{M}_{i,l} = -\hat{x} E_{z_{i,l}} \delta(y) \alpha_{i,l}(x) \delta(z - z') \qquad (40)$$

Using Equations 32, 35 and 37, the y component of the magnetic field is:

$$H_{x_{i,l}} = -E_{z_{i,l}} \sum_n c_{n_{i,l}} h_{x_n}(x,y) e^{-j\beta_n |z - z'|} \qquad (41)$$

where:

$$c_{n_{i,l}} = \frac{1}{4P_n} \int_x h_{x_n}(x',0) \alpha_{i,l}(x') dx' \qquad (42)$$

Now, integrating over z':

$$H_{x_{i,l}}(x,y,z) = -E_{z_{i,l}} \sum_n c_{n_{i,l}} h_{x_m}(x,y) \int_{z'} \lambda_i(z') e^{-j\beta_n |z - z'|} dz' \qquad (43)$$

We will use Galerkin's approach and let the weighting function:

$$\phi_m(x,z) = \phi_{p,r}(x,z) = \alpha_{p,r}(x) \lambda_p(z) \qquad (44)$$

Multiplying Equation 43 by Equation 44 and integrating over y and z, we find:

$$\left\langle \phi_{p,r} \left| H_x \right| \psi_{i,l} \right\rangle = -\sum_n 4P_n c_{n_{p,r}} c_{n_{i,l}} \int_z \lambda_p(z) \int_{z'} \lambda_i(z') e^{-j\beta_n |z - z'|} dz' dz \qquad (45)$$

The double integral in Equation 45 will be defined as:

$$W_i f_{n_{i,p}} = \int_z \lambda_p(z) \int_{z'} \lambda_i(z') e^{-j\beta_n |z-z'|} dz'\,dz \tag{46}$$

where Wi is the width of slot **i** in the z direction. Equation 45 becomes:

$$\left\langle \phi_{p,r} \middle| H_x \middle| \psi_{i,l} \right\rangle = -4W_i \sum_n P_n c_{n_{p,r}} c_{n_{i,l}} f_{n_{i,p}} \tag{47}$$

If the slots are much narrower than a wavelength in the z direction, we can use the step function for the λ function.

$$\lambda_i(z) = 1 \qquad \text{for} \qquad |z - z_i| < \frac{W_i}{2}$$

$$\lambda_i(z) = 0 \qquad \text{for} \qquad |z - z_i| > \frac{W_i}{2} \tag{48}$$

For p• i, Equation 46 becomes:

$$f_{n_{i,p}} = W_p \mathrm{Sa}\left(\frac{\beta_n W_i}{2}\right) \mathrm{Sa}\left(\frac{\beta_n W_p}{2}\right) e^{-j\beta_n |z_i - z_p|} \tag{49}$$

For p=i:

$$f_{n_{i,p}} = W_i \left(\frac{1 - e^{-j\frac{\beta_n W_i}{2}} \mathrm{Sa}\left(\frac{\beta_n W_i}{2}\right)}{j\frac{\beta_n W_i}{2}} \right) \tag{50}$$

Assume that an infinitely narrow (in the y direction) resistor is placed across slot I. The conductance density for the resistor is:

$$g_i(x,z) = \frac{W_i}{R_i} \delta\left(x - x_{R_i}\right) \lambda_i(z) \tag{51}$$

Because λ_i and λ_p only overlap when i=p, the matrix element due to the resistor is:

$$\frac{\left\langle \phi_{p,r} \middle| g \middle| \psi_{i,l} \right\rangle}{W_i} = \frac{\delta_{i,p}}{R_i} \alpha_{p,r}\left(x_{R_i}\right) \alpha_{i,l}\left(x_{R_i}\right) \tag{52}$$

<u>Relativistic Beam Current In A Waveguide</u>

Because of it's high energy, a relativistic beam can be thought of as a current source. A small piece of the beam located at x_b, y_b, has a current density of the form:

$$\vec{J}(x,y,z) = \frac{i_b}{2} \delta(x - x_b) \delta(y - y_b) \delta(z - z') \tag{53}$$

The factor of ½ is results from exploiting the method of images. The y component of magnetic field in the waveguide can be found using Equations 32, 35-37:

$$H_x(x,y) = -\frac{i_b}{2}\sum_n \frac{\hat{e}_{z_n}(x_b, y_b)}{4P_n} h_{x_n}(x,y) e^{-j\beta_n|z-z'|} \tag{54}$$

Equation 54 must be multiplied by the z dependence of the beam current density and integrated:

$$H_x^{inc}(x,y,z) = -\frac{i_b}{2}\sum_n \frac{\hat{e}_{z_n}(x_b, y_b)}{4P_n} h_{x_n}(x,y) \int_{-\infty}^{\infty} e^{-j\kappa z'} e^{-j\beta_n|z-z'|} dz' \tag{55}$$

where:

$$\kappa = \frac{\omega}{c} \tag{56}$$

Equation 55 becomes:

$$H_x^{inc}(x,y,z) = -\frac{i_b}{2}\sum_n \frac{\hat{e}_{z_n}(x_b, y_b)}{4P_n} \frac{2j\beta_n}{\kappa^2 - \beta_n{}^2} h_{x_n}(x,y) e^{-j\kappa z} \tag{57}$$

The matrix element on the left hand side of Equation 31 is found by evaluating Equation 57 at y=0 and multiplying the result by $\phi_{p,r}$ and integrating:

$$\left\langle \phi_{p,r} \middle| H_x^{inc} \right\rangle = -\frac{i_b}{2} W_p \mathrm{Sa}\left(\frac{\kappa W_p}{2}\right) e^{-j\kappa z_p} \sum_n c_{n_{p,r}} \hat{e}_{z_n}(x_b, y_b) \frac{2j\beta_n}{\beta_n{}^2 - \kappa^2} \tag{58}$$

<u>Waveguide Mode as the Incident Field</u>

Consider the case where the incident field is a waveguide mode.

$$H_x^{inc}(x,y,z) = \hat{h}_{xn0}(x,y) e^{-j\beta_{n0} z} \tag{59}$$

Using Equation 44, the left hand side matrix element is:

$$\left\langle \phi_{p,r} \middle| H_x^{inc} \right\rangle = \left(\int_z e^{-j\beta_{n0} z} \lambda_p(z) dz \right) \left(\int_x \hat{h}_{xn0}(x,0) \alpha_{p,r}(x) dx \right) \tag{60}$$

Using Equation42 Equation 60 becomes:

$$\left\langle \phi_{p,r} \middle| H_x^{inc} \right\rangle = 4 W_p P_{n0} c_{n0_{p,r}} \mathrm{Sa}\left(\frac{\beta_{n0} W_p}{2}\right) e^{-j\beta_{n0} z_p} \tag{61}$$

<u>Mode Power</u>

Once Equation 31 is solved, the coupling coefficients for the waveguide modes shown in Equation 32 can be determined. The magnetic current source for the lower waveguide of Figures. 14 due to the electric field in the slots is:

$$\bar{M} = \hat{x}\delta(y)\sum_i\left(\lambda_i(z)\sum_l \alpha_{i,l}(x)E_{z_{i,l}}\right) \tag{62}$$

Substituting Equation 62 into Equation 35, the forward and reverse coupling coefficients are:

$$C_m^{\pm}\Big|_{upper} = \mp\sum_i\left(\mathrm{Sa}\left(\frac{\beta_m W_i}{2}\right)e^{\pm j\beta_m z_i}\sum_l c_{m_{i,l}}\Big|_{upper} W_i E_{z_{i,l}}\right)$$

(63)

$$C_m^{\pm}\Big|_{lower} = \pm\sum_i\left(\mathrm{Sa}\left(\frac{\beta_m W_i}{2}\right)e^{\pm j\beta_m z_i}\sum_l c_{m_{i,l}}\Big|_{lower} W_i E_{z_{i,l}}\right)$$

The power in mode **m** is given by the Poynting Vector:

$$P_m^{\pm} = \frac{1}{2}\mathrm{Re}\left\{\iint_{S_t}\left(\bar{E}_{t_m} \times \bar{H}_{t_m}^*\right)\bullet \hat{z}dS\right\} \tag{64}$$

Using Equations 32 and 36, Equation 64 reduces to:

$$P_m^{\pm} = \left|C_m^{\pm}\right|^2 \mathrm{Re}\{P_m\} \tag{65}$$

Pickup Transfer Impedance

This section will develop a definition of transfer impedance that can be used to compare the waveguide design to cooling arrays that are built with conventional pickups. The difference mode power is:

$$\frac{1}{2}P_{\Delta_{total}} = \frac{1}{2}Z_{\Delta_{pu}}^{\pm}\left(\frac{i_b}{2}\right)^2\left(\frac{y}{d/2}\right)^2 \tag{66}$$

where d is the transverse height of the beam pipe and the ½ factor in the front of the right hand side of Equation 66 is because i_b is a peak current (not rms.) The left hand side of Equation 66 is just the power flowing out of one of the waveguides. Using Equations 65 and 66, the impedance of the array becomes:

$$\sqrt{Z_{\Delta_{pu}}^{\pm}} = \frac{C_0^{\pm}\sqrt{2\,\mathrm{Re}\{P_0\}}}{\left(\frac{i_b}{2}\right)\left(\frac{y}{d/2}\right)} \tag{67}$$

The sum mode power is:

$$\frac{1}{2}P_{\Sigma_{total}} = \frac{1}{2}Z_{\Sigma_{pu}}^{\pm}\left(\frac{i_b}{2}\right)^2 \tag{68}$$

The sum mode impedance of the array becomes:

$$\sqrt{Z^{\pm}_{\Sigma_{pu}}} = \frac{C_0^{\pm}\sqrt{2\,\mathrm{Re}\{P_0\}}}{\left(\frac{i_b}{2}\right)} \tag{69}$$

Kicker Transfer Impedance

The definition of the a kicker impedance is given as:

$$P_k = \frac{1}{2}\frac{\left(\frac{\Delta pc}{q}\right)^2}{Z_k} \tag{70}$$

where Δpc is the change of momentum through the kicker (either longitudinal or transverse), q is the charge of the antiproton, and P_k is the total kicker power. For a particle travelling in the +z direction.

$$\Delta\vec{p} = \int_{t_o}^{t_o+\frac{L}{c}} \vec{F}\left(z = c(t-t_o)-\frac{L}{2}, t\right)dt \tag{71}$$

where F(z,t) is the force on the particle and L is the length of the array. Since the time dependence of the force is $e^{j\omega t}$, Equation 71 becomes:

$$\Delta\vec{p}c = \int_{-\frac{L}{2}}^{\frac{L}{2}} \vec{F}(z)e^{j\kappa z}dz \tag{72}$$

where dt=dz/c and have defined t_o=-L/2c. The force on the particle comes from the electromagnetic wave of the kicker:

$$\vec{F} = q\left(\vec{E} + c\left(\hat{z}\times\mu\vec{H}\right)\right) \tag{73}$$

The change in longitudinal momentum becomes:

$$\frac{\Delta pc|_z}{q} = \int_{-\infty}^{\infty} E_z e^{j\kappa z}dz \tag{74}$$

The change in the transverse momentum becomes:

$$\frac{\Delta pc|_y}{q} = \int_{-\infty}^{\infty} \left(E_y + \eta H_x\right)e^{j\kappa z}dz \tag{75}$$

For a particle travelling in the –z direction:

$$\frac{\Delta pc|_z}{q} = \int_{-\infty}^{\infty} E_z e^{-j\kappa z} dz \tag{76}$$

$$\frac{\Delta pc|_y}{q} = \int_{-\infty}^{\infty} \left(E_y - \eta H_x\right) e^{-j\kappa z} dz \tag{77}$$

The integrals in Equations 74-77 can be evaluated by using reciprocity. The reciprocity law stated in Equation 33 can be rewritten as:

$$\oint\!\!\oint_{S_o} \left(\vec{E}^p \times \vec{H}^k - \vec{E}^k \times \vec{H}^p\right) \bullet \hat{n} dS = \iiint_V \left(\vec{H}^k \bullet \vec{M}^p - \vec{E}^k \bullet \vec{J}^p\right) dv \tag{78}$$

where **p** designates the pickup fields and sources, and **k** designates the kicker fields (there are no kicker sources.) The geometry that defines the surface $\mathbf{S_o}$ and the volume **V** is shown in Figure 16.

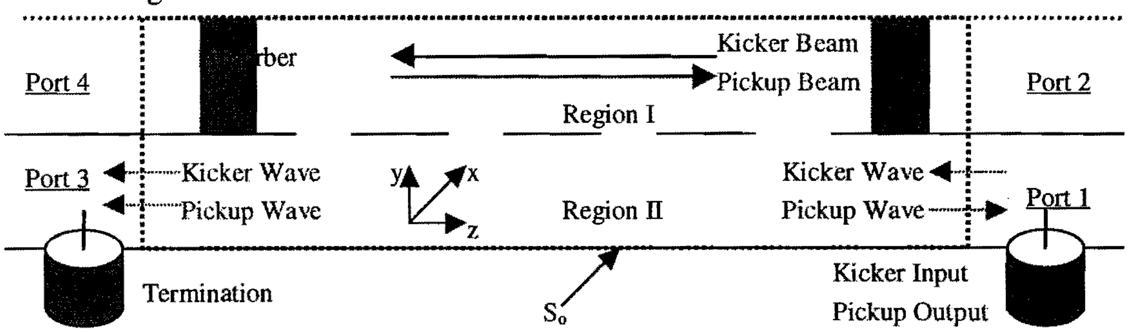

Figure 16. Waveguide pickup showing surface for reciprocity integral

For a kicker, the fields at the 4 ports are:

$$\vec{E}_1^k = \alpha_k \hat{e}_{t_0}\left(S_{1,1}+1\right) \qquad \vec{E}_2^k = 0 \qquad \vec{E}_3^k = \alpha_k \hat{e}_{t_0} S_{3,1} \qquad \vec{E}_4^k = 0$$

$$\vec{H}_1^k = \alpha_k \hat{h}_{t_0}\left(S_{1,1}-1\right) \qquad \vec{H}_2^k = 0 \qquad \vec{H}_3^k = -\alpha_k \hat{h}_{t_0} S_{3,1} \qquad \vec{H}_4^k = 0$$

$$\vec{E}_1^p = \alpha_{p_f} \hat{e}_{t_0} \qquad \vec{E}_2^p = \vec{E}_b e^{-j\theta} \qquad \vec{E}_3^p = \alpha_{p_r} \hat{e}_{t_0} \qquad \vec{E}_4^p = \vec{E}_b e^{j\theta}$$

$$\vec{H}_1^p = \alpha_{p_f} \hat{h}_{t_0} \qquad \vec{H}_2^p = \vec{H}_b e^{-j\theta} \qquad \vec{H}_3^p = -\alpha_{p_r} \hat{h}_{t_0} \qquad \vec{H}_4^p = \vec{H}_b e^{j\theta}$$

(79)

where E_b and H_b are the beam fields in an unperturbed beam pipe and $S_{1,1}$ and $S_{3,1}$ are the scattering parameters of the kicker. It was assumed that the absorber kills the pickup and kicker fields at ports 2 and 4 and that only the fundamental mode propagates in ports 1 and 3. The left hand side of the reciprocity integral of Equation 78 is zero for ports 2, 3, and 4. The integral is non-zero only at port 1. Equation 78 becomes:

$$4P_0 \alpha_k \alpha_{p_f} = \iiint_V \left(\vec{E}^k \bullet \vec{J}^p - \vec{H}^k \bullet \vec{M}^p\right) dv \tag{80}$$

If the beam current density is chosen to be:

$$\vec{J}^p = \hat{z}\frac{i_b}{2}\delta(x - x_b)\delta(y - y_b)e^{-j\kappa z} \tag{81}$$

which is the description of a beam travelling in the +z direction. Equation 80 becomes:

$$4P_0\alpha_k\alpha_{p_f} = \frac{i_b}{2}\int_{-\infty}^{\infty} E_z^k(x_b, y_b)e^{-j\kappa z}dz \tag{82}$$

Using Equations 70 and 76,

$$4P_0\alpha_k\alpha_{p_f} = \frac{i_b}{2}\sqrt{2Z_{\Sigma_k}P_k} \tag{83}$$

Since the total kicker power is equal to the sum of the power in the upper and lower waveguides of Figure 2:

$$\frac{1}{2}P_k = \alpha_k{}^2 P_0 \tag{84}$$

The pickup coefficient in Equation 83 can be found from Equation 69. Equation 83 becomes:

$$2Z_{\Sigma_p} = Z_{\Sigma_k} \tag{85}$$

For the transverse case, we need two transverse current sources:

$$\vec{J}^p = \hat{y}\frac{i_b}{2}\delta(x - x_b)\delta(y - y_b)e^{-j\kappa z} \tag{86}$$

$$\vec{M}^p = \hat{x}\eta\frac{i_b}{2}\delta(x - x_b)\delta(y - y_b)e^{-j\kappa z} \tag{87}$$

Also define:

$$\alpha_{p_f}{}^2 P_0 = \frac{1}{2}\left(\frac{i_b}{2}\right)^2 \tilde{Z}_{\Delta_p} \tag{88}$$

Note that the impedance in Equation 80 is not the same impedance as defined in Equation 66. Substituting Equations 84, 86-88 into Equation 80 and using the definition of the kicker impedance found in Equations 70 and 77:

$$2\tilde{Z}_{\Delta_p} = Z_{\Delta_k} \tag{89}$$

To solve for the vector element on the left hand side of Equation 31, the same procedure that was developed in Equations 53-58 is used. First, the Green's function for the following sources is found:

$$\vec{J}^p = \hat{y}\frac{i_b}{2}\delta(x - x_b)\delta(y - y_b)\delta(z - z') \tag{90}$$

$$\vec{M}^p = \hat{x}\eta \frac{i_b}{2}\delta(x - x_b)\delta(y - y_b)\delta(z - z') \tag{91}$$

Using Equation 37:

$$C_m^{J\pm} = \frac{1}{4P_m}\frac{i_b}{2}\hat{e}_{y_m}(x_b, y_b)e^{\pm j\beta_m z'} \tag{92}$$

$$C_m^{M\pm} = \pm\frac{1}{4P_m}\eta\frac{i_b}{2}\hat{h}_{x_m}(x_b, y_b)e^{\pm j\beta_m z'} \tag{93}$$

where the J,M superscripts indicate whether the coefficient is for the electric or magnetic current source, respectively. The plus/minus sign in the superscript indicates whether the solution is for z>z' and z<z', respectively. The transverse magnetic field in a waveguide is proportional to the transverse electric field:

$$\hat{h}_{x_m} = -Z_m^{wave}\hat{e}_{y_m} \tag{94}$$

Equation 93 becomes:

$$C_m^{M\pm} = \mp\frac{1}{4P_m}\frac{\eta}{Z_m^{wave}}\frac{i_b}{2}\hat{e}_{y_m}(x_b, y_b)e^{\pm j\beta_m z'} \tag{95}$$

The magnetic fields for the point current sources are:

$$H_{x_m}^{J\pm}(x, y, z, z') = \pm\frac{1}{4P_m}\frac{i_b}{2}\hat{e}_{y_m}(x_b, y_b)\hat{h}_{x_m}(x, y)e^{-j\beta_m|z-z'|} \tag{96}$$

$$H_{x_m}^{M\pm}(x, y, z, z') = -\frac{1}{4P_m}\frac{\eta}{Z_m^{wave}}\frac{i_b}{2}\hat{e}_{y_m}(x_b, y_b)\hat{h}_{x_m}(x, y)e^{-j\beta_m|z-z'|} \tag{97}$$

The magnetic field due to the extended electric beam source of Equation 86 is found by integrating Equation 96:

$$H_{x_m}^{J}(x, y, z) = \int_{-\infty}^{z} H_{x_m}^{J+}(x, y, z, z')e^{-j\kappa z'}dz' + \int_{z}^{\infty} H_{x_m}^{J-}(x, y, z, z')e^{-j\kappa z'}dz' \tag{98}$$

$$\begin{aligned} H_{x_m}^{J}(x, y, z) &= \frac{1}{4P_m}\frac{i_b}{2}\hat{e}_{y_m}(x_b, y_b)\hat{h}_{x_m}(x, y) \\ &\cdot\left[e^{-j\beta_m z}\int_{-\infty}^{z} e^{j(\beta_m - \kappa)z'}dz' - e^{j\beta_m z}\int_{z}^{\infty} e^{-j(\beta_m + \kappa)z'}dz'\right] \end{aligned} \tag{99}$$

Using the radiation condition, the integrals vanish at infinity. Equation 99 becomes:

$$H_{x_m}^{J}(x, y, z) = \frac{1}{4P_m}\frac{i_b}{2}\hat{e}_{y_m}(x_b, y_b)\frac{2j\kappa}{\kappa^2 - \beta_m{}^2}\hat{h}_{x_m}(x, y)e^{-j\kappa z} \tag{100}$$

The magnetic field due to the magnetic current is:

$$H^{M}_{x_m}(x,y,z) = -\frac{1}{4P_m}\frac{i_b}{2}\hat{e}_{y_m}(x_b,y_b)\frac{\eta}{Z_m^{wave}}\frac{2j\beta_m}{\kappa^2-\beta_m{}^2}\hat{h}_{x_m}(x,y)e^{-j\kappa z} \tag{101}$$

The total magnetic field is the sum of Equations 100 and 101:

$$H_{x_m}(x,y,z) = \frac{j}{2P_m}\frac{i_b}{2}\hat{e}_{y_m}(x_b,y_b)\frac{\kappa - \frac{\eta}{Z_m^{wave}}\beta_m}{\kappa^2-\beta_m{}^2}\hat{h}_{x_m}(x,y)e^{-j\kappa z} \tag{102}$$

The matrix element on the left hand side of Equation 31 is found by evaluating Equation 102 at y=0, summing the result over all the **m** modes, multiplying this result by $\phi_{p,r}$ and integrating over x and z:

$$\left\langle \phi_{p,r} \middle| H_x^{inc} \right\rangle = 2j\frac{i_b}{2}W_p Sa\left(\frac{\kappa W_p}{2}\right)e^{-j\kappa z_p}\sum_m c_{m_{p,r}}\hat{e}_{y_m}(x_b,y_b)\frac{\kappa - \frac{\eta}{Z_m^{wave}}\beta_m}{\kappa^2-\beta_m{}^2} \tag{103}$$

<u>Summary of Equations</u>

The following matrix equation is to be solved for the electric field in the slots.

$$\left\langle \phi_m \middle| H_x^{(inc)} \right\rangle = \sum_n \left(\sum_k \langle \phi_m | \boldsymbol{H}_x^{(k)} | \psi_n \rangle + \langle \phi_m | g | \psi_n \rangle \right) E_{z_n} \tag{104}$$

where ψ_n (and ϕ_m) are expansion functions for the field in the slot:

$$\psi_n(x,z) = \psi_{i,l}(x,z) = \alpha_{i,l}(x)\lambda_i(z)$$

$$\lambda_i(z) = 1 \qquad \text{for} \qquad |z - z_i| < \frac{W_i}{2}$$

$$\lambda_i(z) = 0 \qquad \text{for} \qquad |z - z_i| > \frac{W_i}{2} \tag{105}$$

The right hand side matrix elements are:

$$\frac{\langle \phi_{p,r} | g | \psi_{i,l} \rangle}{W_i} = \frac{\delta_{i,p}}{R_i}\alpha_{p,r}(x_{R_i})\alpha_{i,l}(x_{R_i}) \tag{106}$$

$$\left\langle \phi_{p,r} \middle| \boldsymbol{H}_x \middle| \psi_{i,l} \right\rangle = -4W_i \sum_n P_n c_{n_{p,r}} c_{n_{i,l}} f_{n_{i,p}} \tag{107}$$

where:

$$P_m = \frac{1}{2}\iint_{S_t}\left(\hat{e}_{t_m} \times \hat{h}_{t_m}\right) \bullet \hat{z}\, dS \tag{108}$$

$$c_{n_{i,l}} = \frac{1}{4P_n} \int_x h_{x_n}(x',0)\alpha_{i,l}(x')dx' \qquad (109)$$

For p• i,:

$$f_{n_{i,p}} = W_p \mathrm{Sa}\left(\frac{\beta_n W_i}{2}\right) \mathrm{Sa}\left(\frac{\beta_n W_p}{2}\right) e^{-j\beta_n |z_i - z_p|} \qquad (110)$$

For p=i:

$$f_{n_{i,p}} = W_i \left(\frac{1 - e^{-j\frac{\beta_n W_i}{2}} \mathrm{Sa}\left(\frac{\beta_n W_i}{2}\right)}{j\frac{\beta_n W_i}{2}} \right) \qquad (111)$$

For a difference mode pickup, a sum mode pickup, and a sum mode kicker, the vector elements on the left hand side of Equation 104 are:

$$\left\langle \phi_{p,r} \middle| H_x^{inc} \right\rangle = -\frac{i_b}{2} W_p \mathrm{Sa}\left(\frac{\kappa W_p}{2}\right) e^{-j\kappa z_p} \sum_n c_{n_{p,r}} \hat{e}_{z_n}(x_b, y_b) \frac{2j\beta_n}{\beta_n{}^2 - \kappa^2} \qquad (112)$$

For a difference mode kicker, the vector elements are:

$$\left\langle \phi_{p,r} \middle| H_x^{inc} \right\rangle = 2j\frac{i_b}{2} W_p \mathrm{Sa}\left(\frac{\kappa W_p}{2}\right) e^{-j\kappa z_p} \sum_m c_{m_{p,r}} \hat{e}_{y_m}(x_b, y_b) \frac{\kappa - \frac{\eta}{Z_m^{wave}} \beta_m}{\kappa^2 - \beta_m{}^2} \qquad (113)$$

Once the matrix equation Equation 104 is inverted for the electric field in the slots, the amplitude of the waveguide modes flowing out of the structure can be calculated. The mode coefficients for the output (lower) waveguide is:

$$C_m^{\pm}\Big|_{lower} = \pm \sum_i \left(\mathrm{Sa}\left(\frac{\beta_m W_i}{2}\right) e^{\pm j\beta_m z_i} \sum_l c_{m_{i,l}}\Big|_{lower} W_i E_{z_{i,l}} \right) \qquad (114)$$

where the top sign is for the signal traveling in the + z direction (for a beam traveling in the +z direction) and the bottom sign is for the signal travelling in the –z direction. Note that this formula is only valid for the regions upstream and downstream of the slots (not inside the slot region.)

The sum mode pickup impedance is defined as:

$$P_{\Sigma_{pu_{total}}} = \left(\frac{i_b}{2}\right)^2 Z_{\Sigma_{pu}} \qquad (115)$$

where i_b is a **peak** current (not rms.) The sum mode pickup impedance is given from the power travelling in the fundamental waveguide mode:

$$\sqrt{Z^{\pm}_{\Sigma_{pu}}} = \frac{C_0^{\pm}\sqrt{2P_0}}{\left(\frac{i_b}{2}\right)} \tag{116}$$

For the difference mode pickup, the impedance is defined as:

$$P_{\Delta_{pu_{total}}} = \left(\frac{i_b}{2}\right)^2 \left(\frac{y}{d/2}\right)^2 Z_{\Delta_{pu}} \tag{117}$$

which results in:

$$\sqrt{Z^{\pm}_{\Delta_{pu}}} = \frac{C_0^{\pm}\sqrt{2P_0}}{\left(\frac{i_b}{2}\right)\left(\frac{y}{d/2}\right)} \tag{118}$$

A kicker impedance is defined as:

$$P_k = \frac{1}{2}\frac{\left(\frac{\Delta pc}{q}\right)^2}{Z_k} \tag{119}$$

From reciprocity, the sum mode kicker impedance is:

$$2Z_{\Sigma_p} = Z_{\Sigma_k} \tag{120}$$

where the signal and beam directions shown in Figure 16 is followed. For the difference mode kicker, the electric field in the slots and the hence the mode coefficients is calculated using the vector described by Equation 113. The kicker impedance is becomes

$$\sqrt{Z^{\pm}_{\Delta_k}} = \frac{2C_0^{\pm}\sqrt{P_0}}{\left(\frac{i_b}{2}\right)} \tag{121}$$

Waveguide Modes For A Rectangular Waveguide

We will assume an electric vector potential of the form:

$$\vec{F} = \hat{x}F_{x_{m,n}}(x,y)e^{-j\beta_{m,n}z} \tag{122}$$

Equations 5 and 7 become:

$$\hat{e}(x,y)_x = 0 \tag{123a}$$

$$\hat{e}(x,y)_y = j\beta_{m,n}\eta c F_{x_{m,n}} \tag{123b}$$

$$\hat{e}(x,y)_z = \eta c \frac{\partial F_{x_{m,n}}}{\partial y} \tag{123c}$$

$$\hat{h}(x,y)_x = -\frac{j\omega}{\kappa^2}\left(\kappa^2 + \frac{\partial^2}{\partial x^2}\right)F_x \tag{123d}$$

$$\hat{h}(x,y)_y = -\frac{j\omega}{\kappa^2}\frac{\partial^2 F_x}{\partial x \partial y} \tag{123e}$$

$$\hat{h}(x,y)_z = -\frac{\omega\beta_{m,n}}{\kappa^2}\frac{\partial F_x}{\partial x} \tag{123f}$$

The boundary conditions require that:

$$\vec{E}_y = 0 \text{ at } x = \pm a/2 \tag{124a}$$

$$\vec{E}_z = 0 \text{ at } y = 0, b \text{ and at } x = \pm a/2 \tag{124b}$$

For simplicity, we will consider the beam centered at x=0. This will require F_x to be even in x. The electric vector potential that satisfies these equations and constraints is:

$$F_{x_{m,n}} = \frac{1}{j\beta_{m,n}\eta c} E_o \cos\left(\frac{(2m+1)\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \tag{125}$$

$$\beta_{m,n}^2 = \kappa^2 - \left(\frac{(2m+1)\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 \tag{126}$$

Equation 67 becomes:

$$\hat{e}_{x_{m,n}} = 0 \tag{127a}$$

$$\hat{e}_{y_{m,n}} = E_o \cos\left(\frac{(2m+1)\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \tag{127b}$$

$$\hat{e}_{z_{m,n}} = -\frac{1}{j\beta_{m,n}}\frac{n\pi}{b}E_o \cos\left(\frac{(2m+1)\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \tag{127c}$$

$$\hat{h}_{x_{m,n}} = -\frac{1}{\eta\kappa\beta_{m,n}}\left(\kappa^2 - \left(\frac{(2m+1)\pi}{a}\right)^2\right)E_o \cos\left(\frac{(2m+1)\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \tag{127d}$$

$$\hat{h}_{y_{m,n}} = -\frac{1}{\eta\kappa\beta_{m,n}}\left(\frac{(2m+1)\pi}{a}\right)\left(\frac{n\pi}{b}\right)E_o \sin\left(\frac{(2m+1)\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \tag{127e}$$

$$\hat{h}_{z_{m,n}} = -\frac{j}{\eta\kappa}\left(\frac{(2m+1)\pi}{a}\right)E_o \sin\left(\frac{(2m+1)\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \tag{127f}$$

Equation 36 becomes:

$$P_{m,n} = \frac{1}{2}\frac{(E_o b)^2}{2d_n\eta\frac{b}{a}\frac{\kappa\beta_{m,n}}{\left(\kappa^2-\left(\frac{(2m+1)\pi}{a}\right)^2\right)}} \tag{128}$$

where d_n=1 for n=0 and d_n =2 for n• 0. The denominator of Equation 128 can be considered the power impedance of the waveguide.
The wave impedance is:

$$Z_{m,n}^{wave} = \eta\frac{\kappa\beta_{m,n}}{\left(\kappa^2-\left(\frac{(2m+1)\pi}{a}\right)^2\right)} \tag{129}$$

so that the power impedance is:

$$Z_{m,n}^{power} = 2d_n\frac{b}{a}Z_{m,n}^{wave} \tag{130}$$

Because we are considering only even modes in x and that the z component of electric field must vanish at the ends of the slot ($x = \pm L_i/2$), a reasonable expansion function for $\alpha(x)$ is:

$$\alpha_{i,l}(x) = \cos\left(\frac{(2l+1)\pi}{L_i}x\right) \tag{131}$$

Equation 42 becomes

$$c_{n_{i,l}} = \frac{1}{E_o}\frac{d_n}{ab}\frac{2L_i}{\pi}\frac{2l+1}{(2m+1)^2\left(\frac{L_i}{a}\right)^2-(2l+1)^2}\cos\left((2m+1)\frac{\pi}{2}\frac{L_i}{a}+l\pi\right) \tag{133}$$

<u>Calculation of the electric field along the length of the waveguides.</u>

We can assume that each one of the slots in Figure 13 to be a magnetic current source. The total field in the waveguide will be a sum of the fields resulting from each slot. For a single slot and slot mode, the magnetic current source is:

$$\vec{M}_{i,l} = \hat{x}E_{z_{i,l}}\alpha_{i,l}(x)\delta(y)\lambda_i(z) \tag{134}$$

Using Equations 37, the waveguide mode coupling coefficients are:

$$C^{\pm}_{n_{i,l}} = \pm E_{z_{i,l}} \frac{1}{4P_n} \int_x \hat{h}_{x_n}(x',0)\alpha_{i,l}(x')dx' \int_z \lambda_i(z') e^{\pm j\beta_n z'} dz' \tag{135}$$

If the observation point is at the center of the waveguide (x=0) then Equation 135 becomes:

$$C^{\pm}_{n_{i,l}} = \pm E_{z_{i,l}} c_{n_{i,l}} \int_z \lambda_i(z') e^{\pm j\beta_n z'} dz' \tag{136}$$

For: $z < z_i - \frac{W_i}{2}$ $\quad C^{+}_{n_{i,l}} = 0$

Equation 136 becomes:

$$C^{-}_{n_{i,l}} = -E_{z_{i,l}} c_{n_{i,l}} \int_{z_i - \frac{W_i}{2}}^{z_i + \frac{W_i}{2}} e^{-j\beta_n z'} dz' \tag{137}$$

$$C^{-}_{n_{i,l}} = -W_i E_{z_{i,l}} c_{n_{i,l}} Sa\left(\frac{\beta_n W_i}{2}\right) e^{-j\beta_n z_i} \tag{138}$$

The electric field is found from Equation 32:

$$\vec{E}_{i,l} = -W_i E_{z_{i,l}} \sum_n c_{n_{i,l}} Sa\left(\frac{\beta_n W_i}{2}\right)\left(\hat{e}_{y_n} - \hat{e}_{z_n}\right) e^{j\beta_n (z - z_i)} \tag{139}$$

For: $z > z_i + \frac{W_i}{2}$ $\quad C^{-}_{n_{i,l}} = 0$

The electric field is:

$$\vec{E}_{i,l} = W_i E_{z_{i,l}} \sum_n c_{n_{i,l}} Sa\left(\frac{\beta_n W_i}{2}\right)\left(\hat{e}_{y_n} + \hat{e}_{z_n}\right) e^{-j\beta_n (z - z_i)} \tag{140}$$

For: $\left(z_i - \frac{W_i}{2}\right) < z < \left(z_i + \frac{W_i}{2}\right)$

$$C^{+}_{n_{i,l}} = E_{z_{i,l}} c_{n_{i,l}} \int_{z_i - \frac{W_i}{2}}^{z} e^{j\beta_n z'} dz' \tag{141}$$

$$C^{-}_{n_{i,l}} = -E_{z_{i,l}} c_{n_{i,l}} \int_{z}^{z_i + \frac{W_i}{2}} e^{-j\beta_n z'} dz' \tag{142}$$

The electric field for the forward and reverse waves at x=0 is:

$$\vec{E}_{i,l}^{+} = W_i E_{z_{i,l}} \sum_n c_{n_{i,l}} \left(\hat{e}_{y_n} + \hat{e}_{z_n}\right) \left(\frac{1 - e^{-j\frac{\beta_n W_i}{2}} e^{-j\beta_n (z-z_i)}}{j\beta_n W_i} \right) \tag{143}$$

$$\vec{E}_{i,l}^{-} = -W_i E_{z_{i,l}} \sum_n c_{n_{i,l}} \left(\hat{e}_{y_n} - \hat{e}_{z_n}\right) \left(\frac{1 - e^{-j\frac{\beta_n W_i}{2}} e^{j\beta_n (z-z_i)}}{j\beta_n W_i} \right) \tag{144}$$

The total electric field for the region enclosed in the slot is the sum of Equations 143 and 144:

$$\vec{E}_{i,l} \bullet \hat{y} = W_i E_{z_{i,l}} \sum_n c_{n_{i,l}} \left(\hat{e}_{y_n} \bullet \hat{y}\right) \frac{e^{-j\frac{\beta_n W_i}{2}}}{\frac{\beta_n W_i}{2}} \sin\left(\beta_n (z - z_i)\right) \tag{145}$$

$$\vec{E}_{i,l} \bullet \hat{z} = W_i E_{z_{i,l}} \sum_n c_{n_{i,l}} \left(\hat{e}_{z_n} \bullet \hat{z}\right) \frac{1}{j\frac{\beta_n W_i}{2}} \left(1 - e^{-j\frac{\beta_n W_i}{2}} \cos\left(\beta_n (z - z_i)\right) \right) \tag{146}$$

In summary:

For: $z < z_i - \frac{W_i}{2}$

$$\vec{E}_{i,l} \bullet \hat{y} = -W_i E_{z_{i,l}} \sum_n c_{n_{i,l}} \mathrm{Sa}\left(\frac{\beta_n W_i}{2}\right) \left(\hat{e}_{y_n} \bullet \hat{y}\right) e^{j\beta_n (z-z_i)} \tag{147}$$

$$\vec{E}_{i,l} \bullet \hat{z} = W_i E_{z_{i,l}} \sum_n c_{n_{i,l}} \mathrm{Sa}\left(\frac{\beta_n W_i}{2}\right) \left(\hat{e}_{z_n} \bullet \hat{z}\right) e^{j\beta_n (z-z_i)} \tag{148}$$

For: $\left(z_i - \frac{W_i}{2}\right) < z < \left(z_i + \frac{W_i}{2}\right)$

$$\vec{E}_{i,l} \bullet \hat{y} = W_i E_{z_{i,l}} \sum_n c_{n_{i,l}} \left(\hat{e}_{y_n} \bullet \hat{y}\right) \frac{e^{-j\frac{\beta_n W_i}{2}}}{\frac{\beta_n W_i}{2}} \sin\left(\beta_n (z - z_i)\right) \tag{149}$$

$$\vec{E}_{i,l} \bullet \hat{z} = W_i E_{z_{i,l}} \sum_n c_{n_{i,l}} \left(\hat{e}_{z_n} \bullet \hat{z}\right) \frac{1}{j\frac{\beta_n W_i}{2}} \left(1 - e^{-j\frac{\beta_n W_i}{2}} \cos\left(\beta_n (z - z_i)\right) \right) \tag{150}$$

For: $z > z_i + \frac{W_i}{2}$

$$\vec{E}_{i,l} \bullet \hat{y} = W_i E_{z_{i,l}} \sum_n c_{n_{i,l}} \mathrm{Sa}\left(\frac{\beta_n W_i}{2}\right)\left(\hat{e}_{y_n} \bullet \hat{y}\right) e^{-j\beta_n (z-z_i)} \tag{151}$$

$$\vec{E}_{i,l} \bullet \hat{z} = W_i E_{z_{i,l}} \sum_n c_{n_{i,l}} \mathrm{Sa}\left(\frac{\beta_n W_i}{2}\right)\left(\hat{e}_{z_n} \bullet \hat{z}\right) e^{-j\beta_n (z-z_i)} \tag{152}$$